# U.G. 1st Semester Examination - 2020

# MATHEMATICS

## [HONOURS]

### Course Code : MATH-H-CC-T-02

Full Marks : 60 Time : $2\frac{1}{2}$ Hours

*The figures in the right-hand margin indicate marks.*

*The notations and symbols have their usual meanings.*

1. Answer any **ten** questions: $2\times10=20$

i) Using Descartes' rule of signs, find the nature of roots of the following equation:

$$x^4+15x^2+7x-11=0.$$

ii) State the well-ordering principle. Is the set $\left\{n\in\mathbb{N} \mid n>-\sqrt{2}\right\}$ well-ordered? Justify.

iii) Show that every odd degree polynomial has at least one real root.

iv) Can an eigen vector be zero? Justify your answer.



v) Let $A$ and $B$ be any two square matrices. Is it true that $\text{tr}(AB)=\text{tr}(A).\text{tr}(B)$? Justify your answer.

vi) Use the principle of mathematical induction to prove that

$$1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{4n^3-n}{3},\ \forall n\geq1.$$

vii) What is an inconsistent system of linear equations? How do they arise?

viii) Show that corresponding to every real matrix $A_{m\times n}$ there is a linear transformation $T:\mathbb{R}^n\to\mathbb{R}^m$.

ix) Show that only subspaces of $\mathbb{R}$ are $\{0\}$ and $\mathbb{R}$.

x) Prove or disprove:

$W=\left\{A\in M_n(\mathbb{R}) \mid tr(A)=0\right\}$ is a subspace of $M_n(\mathbb{R})$.

xi) Is the set {(1,2,3), (1,0,–1), (0,2,4)} linearly independent? Justify your answer.

xii) Can a vector space have more than one basis? Justify.

xiii) Solve the equation $x^3 - 3x^2 + 4 = 0$, two of its roots being equal.

xiv) If $x_1$, $x_2$, $x_3$ are three positive numbers such that $x_1 + 2x_2 + 3x_3 = 60$, what is the smallest possible value of the sum $x_1^2 + x_2^2 + x_3^2$?

xv) Consider the equivalence relation ~ on $\mathbb{Z}$ given by $m \sim n$ if and only if $m^2 - n^2$ is a multiple of 5. Find the corresponding partition of $\mathbb{Z}$.

2. Answer any **four** questions: 5×4=20

a) i) Let $V$ be the space of polynomials from $\mathbb{R}$ into $\mathbb{R}$ which have degree less than or equal to 3. Let $D: V \to V$ be the differentiation operator. What is the matrix representation of $D$ relative to the standard ordered basis of $V$. 4

ii) If we take a different basis for $V$, then how the matrix with respect to this new basis will be related to the above matrix? 1

b) Find the condition that the bi-quadratic equation $x^4 + px^3 + qx^2 + rx + s = 0$ should have its roots connected by the relation $\beta + \gamma = \alpha + \delta$.



c) Find all solutions of the following system of linear equations:

$$2x_1 - 3x_2 - 7x_3 + 5x_4 + 2x_5 = -2$$
$$x_1 - 2x_2 - 4x_3 + 3x_4 + x_5 = -2$$
$$2x_1 \qquad - 4x_3 + 2x_4 + x_5 = 3$$
$$x_1 - 5x_2 - 7x_3 + 6x_4 + 2x_5 = -7.$$

d) Find the eigen values and associated eigen vectors of the matrix

$$A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}.$$

e) Find all the equivalence relations on the set $A = \{a, b, c\}$.

f) Let $n$ be any integer. Show that 3 divides one of $n$, $n+1$, $2n+1$.

3. Answer any **two** questions: 10×2=20

a) i) Find the equation whose roots are the squares of the roots of the equation

$$x^4 - x^3 + 2x^2 - x + 1 = 0.$$

Use Descartes' rule of signs to deduce

that given equation has no real roots. 5

ii) Find out all the $n$th roots of unity and show that they all lie in a circle. 5

b) i) Find the rank and nullity of the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 & 6 \\ 2 & -1 & 5 & -1 & 7 \\ -1 & 1 & -4 & 1 & -3 \\ 0 & 1 & -3 & 1 & 1 \end{bmatrix}$$ 5

ii) Is the following matrix invertible?

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}$$

If yes, find out the inverse. 5



c) i) Let $A$ be any finite set and $\phi : A \to A$ is injective. Show that $\phi$ is also surjective. Give an example to show that the this may not be true for infinite set.

3+2=5

ii) Suppose $P(\mathbb{R})$ denotes the space of polynomials over $\mathbb{R}$. Let $T : P(\mathbb{R}) \to P(\mathbb{R})$ be defined as $T(f(x)) = f'(x)$. Check whether $T$ is linear, one-one and onto. Justify your answer.

1+2+2=5

d) i) Use the Cayley-Hamilton theorem to compute $A^{-1}$, where

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 7 \\ 0 & 0 & 9 \end{bmatrix}.$$ 3

ii) Let $A$ be a non-invertible $n \times n$ matrix. Explain why 0 must be an eigenvalue of $A$. Find the geometric multiplicity of the eigenvalue 0 in terms of rank($A$).

2+2=4

iii) Let $A$ be any matrix. Show that $A$ and $A^{-1}$ have same eigen vectors. 3

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