2021
MATHEMATICS
[GENERAL]
Paper : I

Full Marks : 100 Time : 3 Hours

The figures in the right-hand margin indicate marks.
Symbols, notations have their usual meanings.

GROUP-A
(Differential Calculus)
[Marks : 50]

1. Answer any **four** questions: 1×4=4

a) Find the radius of curvature of $\sqrt{x}+\sqrt{y}=1$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$.

b) A monotone sequence is always convergent – Justify.

c) Test the convergence of the series

$$\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\ldots+\left(\frac{n}{2n+1}\right)^n+\ldots.$$

d) Evaluate $\lim_{x\to 0}(1+2x)^{\frac{x+3}{x}}$.

e) Test the differentiability of

$$f(x)=\begin{cases} x+1 , & 0\le x\le 1 \\ 3-x , & 1\le x\le 2, \end{cases}$$

at x=1.

f) State Lagrange's MVT.

2. Answer any **six** questions: 2×6=12

a) Investigate for continuity at $(1, 2)$ of

$$f(x, y)=\begin{cases} x^2+2y , & (x, y)\ne(1, 2) \\ 0 , & (x, y)=(1, 2) \end{cases}.$$

b) Find the value of x for which $(\sin x-\cos x)$ is a maximum or a minimum.

c) Find the pedal equation of the asteroid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

d) If $f(x, y)=|x|+|y|$, show that f is not differentiable at $(0, 0)$.

e) Show that $\log_e(1+x)<x-\frac{x^2}{2(1+x)}$ for $x>0$.

f) If $f(x, y)=\begin{cases} xy & \text{when } |x|\ge|y| \\ -xy & \text{when } |x|<|y| \end{cases}$, show that $f_{xy}(0, 0)\ne f_{yx}(0, 0)$.

g) If $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, will the series $\sum_{n=1}^{\infty} a_{2n}$ be convergent? Justify your answer.

h) Evaluate $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$.

3. Answer any **four** questions: 6×4=24

a) i) Suppose that $x_n \to l$ and $y_n \to m$ as $n \to \infty$, then prove that $x_n + y_n \to l + m$ as $n \to \infty$.

ii) Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{2},\ x_{n+1} = \sqrt{2x_n}$, for $n \ge 1$ converges to 2. 2+4=6

b) i) Show that for $y = x^3 \log x$, $\frac{d^n y}{dx^n} = (-1)^n \frac{6\underline{|n-4}}{x^{n-3}}$.

ii) Find the maximum and minimum values of $f(x) = a\sin^2 x + b\cos^2 x$, where $a > b$.

3+3=6

c) i) State and prove Eulers theoem for homogeneous function in two variable x, y of degree n.

ii) If $f(0) = 0,\ f'(x) = \frac{1}{1+x^2}$, prove without the method of integration that $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$. 3+3=6

d) i) Find all the asymptotes of the curve $y = \frac{3x}{2}\log\left(e - \frac{1}{3x}\right)$.

ii) Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters are connected by $a^2 + b^2 = c^2$. (c being a given constant) 3+3=6

e) i) Test the convergence of the series

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \ldots,\ x > 0.$$

ii) If $y = e^{m\sin^{-1}x}$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$.

3+3=6

f) i) If $f(x, y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & , (x, y) \ne (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$

show that $f_{xy}(0, 0) \ne f_{yx}(0, 0)$.

ii) If $f(x)=\begin{cases}1 & , \quad x \text{ is rational} \\ 0 & , \quad x \text{ is irrational},\end{cases}$

prove that $\lim_{x\to a} f(x)$ does not exist for any real number a. 4+2=6

4. Answer any **one** question: 10×1=10

a) i) If $u=ax^2+2hxy+by^2$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial^2 u}{\partial x \partial y} + \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2}$$

$$= 8\left(ab-h^2\right)u.$$

ii) If

$$f(x, y)=\begin{cases}\left(x^2+y^2\right)\log\left(x^2+y^2\right) & , \quad x^2+y^2 \neq 0 \\ 0 & , \quad x^2+y^2=0\end{cases}$$

show that $f_{xy}(0, 0)=f_{yx}(0, 0)$, although neither f_{xy} nor f_{yx} is continuous at $(0, 0)$. 5+5

b) i) Show that the envelope of a family of circles whose centres lie on the rectangular hyperbola $xy=c^2$ and which pass through the centre of the hyperbola is $\left(x^2+y^2\right)^2=16c^2xy$.

ii) If ρ_1 and ρ_2 be the radii of curvature at the ends of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that

$$\rho_1^{\frac{2}{3}}+\rho_2^{\frac{2}{3}}=\frac{\left(a^2+b^2\right)}{(ab)^{\frac{2}{3}}}.$$ 5+5

GROUP-B
(Integral Calculus)
[Marks : 30]

5. Answer any **four** questions: 2×4=8

a) Find the length of the circumference of the circle $x^2+y^2=25$.

b) Evaluate $\int_{-2}^{2} \frac{x^2 \sin x}{x^6+12}\,dx$.

c) Evalute $\int_0^2 |1-x|\,dx$.

d) Find the value of $\int_0^{\frac{1}{2}} \frac{\sin^{-1} x\,dx}{\sqrt{1-x^2}}$.

e) Show that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

f) Evaluate $\int_0^1 dy \int_0^1 f(x, y)\,dx$, where

$$f(x, y) = \begin{cases} \frac{1}{2} & , \quad y \text{ rational} \\ x & , \quad y \text{ irrational.} \end{cases}$$

6. Answer any **two** questions: 6×2=12

a) i) Obtain a reduction formula for $\int_0^{\frac{\pi}{4}} \tan^n x\, dx$, n being a positive integer ≥ 1 and hence evaluate $\int_0^{\frac{\pi}{4}} \tan^6 x\, dx$.

ii) Evaluate $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^8 x\, dx$. 3+3=6

b) Show that:

i) $\int_0^{\infty} e^{-4x} x^{\frac{3}{2}} dx = \frac{3}{128}\sqrt{\pi}$.

ii) $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right) \ldots \Gamma\left(\frac{8}{9}\right) = \frac{3}{16}\pi^4$. 3+3=6

c) i) Evaluate $\iint_R \sin(x+y)\, dx\, dy$, where $R = \left\{0 \leq x \leq \frac{\pi}{2},\ 0 \leq y \leq \frac{\pi}{2}\right\}$.

ii) Find the volume of the solid generated by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about its base. 3+3=6

7. Answer any **one** question: 10×1=10

a) i) Find the area bounded by $y = 6 + 4x - x^2$ and the chord joining (–2, –6) and (4, 6).

ii) Show that the arc of the upper half of the cardiode $r = a(1 - \cos\theta)$ is bisected at $\theta = \frac{2}{3}\pi$. Also show that the perimeter of the curve is 8a. 5+5=10

b) i) Show that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 + x^3}} < \frac{\pi}{6}$.

ii) Prove that $\iiint (x^2 + y^2 + z^2)\, xyz\, dx\, dy\, dz$ taken throughout the sphere $x^2 + y^2 + z^2 \leq 1$ is zero. 5+5=10

GROUP–C

(Differential Equations)

(Marks : 20)

8. Answer any **two** questions: 1×2=2

a) Construct a differential equation by the elimination of the arbitrary constants a and b from the equation $ax^2 + by^2 = 1$.

b) Find an integrating factor of the differential equation

$$\left(y^2+2x^2y\right)dx+\left(2x^3-xy\right)dy=0.$$

c) Find the order and degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2+\frac{d^2y}{dx^2}+x\sin y=0.$$

9. Answer any **one** question: 2×1=2

a) Solve $\left(\cos y+y\cos x\right)dx+\left(\sin x-x\sin y\right)dy=0$.

b) Find the Particular Integral (P.I.) of the differential equation $\left(D^2-5D-6\right)y=e^{4x}$.

10. Answer any **one** question: 6×1=6

a) Prove that the necessary and sufficient condition that the differential equation

$$M(x,\ y)dx+N(x,\ y)dy=0$$

be exact is $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

b) Find the general and singular solutions of

$$16x^2+2p^2y-p^3x=0.\ \left(p=\frac{dy}{dx}\right).$$

11. Answer any **one** question: 10×1=10

a) i) Solve: $\frac{d^2y}{dx^2}+a^2y=\sec ax$.

ii) Solve: $\frac{dx}{dt}+5x+y=e^t$

$\frac{dy}{dt}+3y-x=e^{2t}$ 5+5

b) i) Find the curve for which the product of the intercepts of the tangent line on the co-ordinate axes is equal to a.

ii) The acceleration of a moving particle being proportional to the cube of its velocity and negative, show that the distance passed over in time t is given by

$$s=\frac{\left\{\sqrt{2kv_0^2t+1}-1\right\}}{kv_0},$$

v_0 being the initial velocity and the distance is measured from the position of the particle at time $t=0$. 10